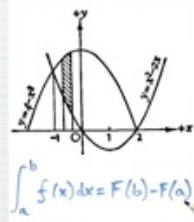
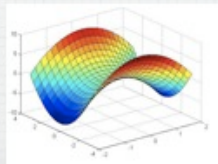


Mathematics and the Theory of Knowledge



17 March 2010



1

Invented or Discovered?



- * What does it mean to say something is discovered? invented?
- * Are there areas of knowledge which are clearly discovered or invented?

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3 Themes

- * Is mathematics invented or discovered?
- * Proof in mathematics and the limitations of proof
- * The "unreasonable effectiveness of mathematics"

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Are mathematical objects real?

- * Consider $i = \sqrt{-1}$
- * What is i ?
- * A solution of the equation $x^2 + 1 = 0$

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Is math created?

- * Why don't we ask if math is "created"?
- * Is this a third category? Are we just re-naming the "invented" category?
- * What do we mean if we say a student is a creative problem-solver?

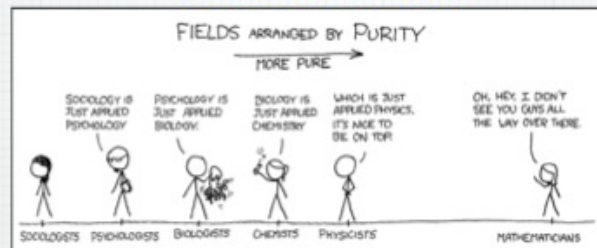
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Axiomatic systems

- * Mathematics epitomizes logic/reasoning as a way of knowing because of its reliance on proof as a way of verifying knowledge.
- * Clearly defined assumptions -- axioms or postulates
- * Conclusions drawn according to defined laws of inference
- * Axioms & definitions \longrightarrow Theorems

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Proof and its limitations



- * Why is this funny? Who are we laughing at?
- * What does "pure" mean in this context?

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Flaws in proofs

- * Is the flaw with the premises or with the reasoning?



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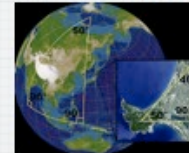
The reality of mathematical objects (again)

- * Could you "invent" a mathematical system where the sum of the angles of a triangle is not 180 degrees?

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Non-Euclidean Geometries

Spherical geometry -- The sum of the angles of a triangle is greater than 180 degrees.



Hyperbolic geometry -- The sum of the angles of a triangle is less than 180 degrees.



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A riddle...

You wake up and see that it's a beautiful clear day and you decide to take a walk.

You walk 5 km south, 5 km east, then 5 km north, and end up back home.

You look out the window and see a bear.

What color is the bear?



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Euclid's Parallel Postulate

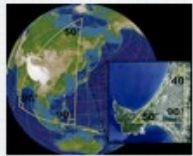
- * Given a line and a point not on the line. There is exactly one line through the given point that is parallel to the given line.



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What if you drop this postulate?

- * Non-Euclidean geometries arise if you don't assume the Parallel Postulate is true.
- * Spherical -- no parallel lines through the given point
- * Hyperbolic -- an infinite number of parallel lines through the given point



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Constructing Pure Mathematics

- * Hilbert
- * Russell/Whitehead

[illegible]


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Constructing pure mathematics

- * We want our formal mathematical systems to be:
- * Consistent -- You can't prove A and Not A are both true.
- * Complete -- Either A or Not A should be provable.

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Formal limitations

- * **Gödel's Incompleteness Theorems (1931):**
 - * **The consistency of any axiomatic system can only be proved in a "larger" system.**
 - * **In any sufficiently rich axiomatic system, there will always be statements that can be neither proved nor refuted.**
- 
- A small, dark, square portrait of Kurt Gödel, showing him from the chest up, wearing a suit and tie.



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Paradoxes

- * Essentially, Gödel proved that any sufficiently rich axiomatic system will always contain paradoxes.



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Math without proof?

What if something just works but you can't prove it?

- * Goldbach conjecture -- Any even number greater than 2 can be written as the sum of two primes.

- * $46 = 17 + 29$

- * $80 = 37 + 43$

Known to be true for even numbers up to 10^{18}

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Four-Color Theorem

- * Any map can be colored with 4 colors or fewer.
- * The proof uses a computer to test special cases.
- * Is a theorem proved with the aid of a computer within the scope of human knowledge?



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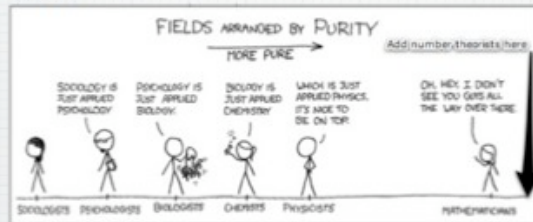
The unreasonable effectiveness of math

- * Why does math do such a good job of mirroring the real world?
 - * Natural and physical sciences
 - * Quantitative methods in the social sciences
 - * And on and on...

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A contemporary example

- * Number theory -- "Mathematics is the queen of the sciences, and number theory is the queen of mathematics."
Gauss



Encryption methods for secure internet transactions are based on advances in number theory.

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Conclusions?

- * Is mathematics invented or discovered?
YES
- * Proof in mathematics and the limitations of proof
Knowing the limitations of proof does not contradict its importance as a way of knowing
- * The "unreasonable effectiveness of mathematics"
Mathematics is both pure and applied.
Don't limit your learning to what you think you need to know.

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For discussion...

- * What critiques does Margaret Wertheim offer of mathematics as an academic discipline?
- * What other points does she raise?

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